
Constructive Learning Modulo Theories

Stefano Teso

DISI, University of Trento, Italy

TESO@DISI.UNITN.IT

Roberto Sebastiani

DISI, University of Trento, Italy

ROBERTO.SEBASTIANI@UNITN.IT

Andrea Passerini

DISI, University of Trento, Italy

PASSERINI@DISI.UNITN.IT

1. Introduction

Modelling problems containing a mixture of Boolean and numerical variables is a long-standing interest of Artificial Intelligence. However, performing inference and learning in hybrid domains is a particularly daunting task. The ability to model these kind of domains is crucial in “learning to design” tasks, that is, learning applications where the goal is to learn from examples how to perform automatic de novo design of novel objects. Apart from hybrid Bayesian Networks, for which efficient inference is limited to conditional Gaussian distributions, there is relatively little previous work on *hybrid* methods. The few existing attempts (Goodman et al., 2008; Wang & Domingos, 2008; Närman et al., 2010; Gutmann et al., 2011; Choi & Amir; Islam et al., 2012) impose strong limitations on the type of constraints they can handle. Inference is typically run by approximate methods, based on variational approximations or sampling strategies. Exact inference, support for hard numeric (in addition to Boolean) constraints and combination of diverse theories are out of the scope of these approaches. In order to overcome these limitations, we focused on the most recent advances in automated reasoning over hybrid domains. Researchers in automated reasoning and formal verification have developed logical languages and reasoning tools that allow for *natively* reasoning over mixtures of Boolean and numerical variables (or even more complex structures). These languages are grouped under the umbrella term of *Satisfiability Modulo Theories* (SMT) (Barrett et al., 2009). Each such language corresponds to a decidable fragment of First-Order Logic augmented with an additional background theory \mathcal{T} , like linear arithmetic over the rationals \mathcal{LRA} or over the integers \mathcal{LIA} . SMT is a decision problem, which consists in find-

Preliminary work. Under review by the Constructive Machine Learning workshop @ ICML 2015. Do not distribute. An extended version of this work was accepted for publication at the Artificial Intelligence Journal.

ing an assignment to both Boolean and theory-specific variables making an SMT formula true. Recently, researchers have leveraged SMT from decision to optimization. The most general framework is that of *Optimization Modulo Theories* (OMT) (Sebastiani & Tomasi, 2015), which consists in finding a *model* for a formula which minimizes the value of some (arithmetical) cost function defined over the variables in the formula.

In this paper we present Learning Modulo Theories (LMT), a max-margin approach for learning in hybrid domains based on Satisfiability Modulo Theories, which allows to combine Boolean reasoning and optimization over continuous linear arithmetical constraints. The main idea is to combine the discriminative power of structured-output SVMs (Tsochantaridis et al., 2005) with the reasoning capabilities of SMT technology. Training structured-output SVMs requires a separation oracle for generating counterexamples and updating the parameters, while an inference oracle is required at prediction stage to generate the highest scoring candidate structure for a certain input. Both tasks can be accomplished by a generalized Satisfiability Modulo Theory solver. We show the potential of LMT on an artificial layout synthesis scenario.

2. LMT as structured-output learning

Structured-output SVMs (Tsochantaridis et al., 2005) generalize max-margin methods to the prediction of output structures by learning a scoring function over joint input-output pairs $f(\mathbf{I}, \mathbf{O}) = \mathbf{w}^T \psi(\mathbf{I}, \mathbf{O})$. Given an input \mathbf{I} , the predicted output will be the one maximizing the scoring function:

$$\mathbf{O}^* = \underset{\mathbf{O}}{\operatorname{argmax}} f(\mathbf{I}, \mathbf{O}) \quad (1)$$

and the problem boils down to finding efficient procedures for computing the maximum. Efficient exact procedures exist for some special cases. In this paper we are inter-

110 ested in the case where inputs and outputs are combina-
 111 tions of discrete and continuous variables, and we'll lever-
 112 age SMT techniques to perform efficient inference in this
 113 setting. Max-margin learning is performed by enforcing a
 114 margin separation between the score of the correct struc-
 115 ture and that of any possible incorrect structure, possibly
 116 accounting for margin violations to be penalized in the ob-
 117 jective function. A cutting plane (CP) algorithm allows to
 118 deal with the exponential number of candidate structures by
 119 iteratively adding the most violated constraint for each ex-
 120 ample pair given the current scoring function, and refining
 121 it by solving the augmented quadratic problem with a stan-
 122 dard SVM solver. The CP algorithm is generic, meaning
 123 that it can be adapted to any structured prediction problem
 124 as long as it is provided with: (i) a joint feature space rep-
 125 resentation ψ of input-output pairs (and consequently a scor-
 126 ing function f); (ii) an oracle to perform inference, i.e. to
 127 solve Equation (1); and (iii) an oracle to retrieve the most
 128 violated constraint of the QP, i.e. to solve the *separation*
 129 problem:

$$\operatorname{argmax}_{\mathcal{O}'} \mathbf{w}^T \psi(\mathbf{I}_i, \mathcal{O}') + \Delta(\mathbf{I}_i, \mathcal{O}_i, \mathcal{O}') \quad (2)$$

134 where $\Delta(\mathbf{I}_i, \mathcal{O}_i, \mathcal{O}')$ is a *loss function* quantifying the
 135 penalty incurred when predicting \mathcal{O}' instead of the correct
 136 output \mathcal{O} . In the following we will detail how these compo-
 137 nents are provided within the LMT framework.

139 **input-output feature map** Recall that in our setting each
 140 object $(\mathbf{I}, \mathcal{O})$ is represented as a set of Boolean and rational
 141 variables:

$$(\mathbf{I}, \mathcal{O}) \in \underbrace{(\{\top, \perp\} \times \dots \times \{\top, \perp\})}_{\text{Boolean part}} \times \underbrace{(\mathbb{Q} \times \dots \times \mathbb{Q})}_{\text{rational part}}$$

146 We indicate Boolean variables using predicates such as
 147 $\text{touching}(i, j)$, and write rational variables as lower-case
 148 letters, e.g. *cost*, *distance*, x , y . Features are represented
 149 in terms of (soft) *constraints* $\{\varphi_k\}_{k=1}^m$, each constraint
 150 φ_k being either a Boolean- or rational-valued function of
 151 the object $(\mathbf{I}, \mathcal{O})$. These constraints are constructed using
 152 the background knowledge available for the domain. For
 153 each Boolean-valued constraint φ_k , we denote its *indicator*
 154 *function* as $\mathbb{1}_k(\mathbf{I}, \mathcal{O})$, which evaluates to 1 if the constraint
 155 is satisfied and to -1 otherwise. Similarly, we refer to the
 156 *cost* of a rational-valued constraint φ_k as $c_k(\mathbf{I}, \mathcal{O}) \in \mathbb{Q}$.
 157 The feature vector $\psi(\mathbf{I}, \mathcal{O})$ is obtained by concatenating
 158 indicator and cost functions of Boolean and rational con-
 159 straints respectively.

161 **inference oracle** Given the feature vector $\psi(\mathbf{I}, \mathcal{O})$, the
 162 scoring function $f(\mathbf{I}, \mathcal{O})$ is a linear combination of indi-
 163 cator and cost functions. Since ψ can be expressed in

165 terms of SMT(\mathcal{LRA}) formulas, the resulting maximiza-
 166 tion problem can be readily cast as an OMT(\mathcal{LRA}) prob-
 167 lem and inference is performed by an OMT-solver (we use
 168 the OPTIMATHSAT solver¹). Given that OMT-solvers are
 169 conceived to *minimize* cost functions rather than maximize
 170 scores, we actually run it on the negated scoring function.

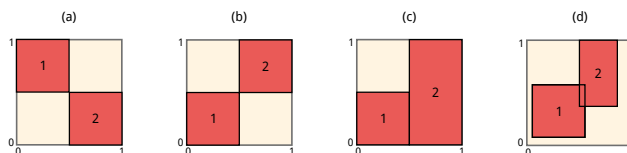
172 **separation oracle** The separation problem amounts at
 173 maximizing the sum of the scoring function and a *loss* func-
 174 tion over output pairs (see eq.(2)). We observe that by pick-
 175 ing a loss function expressible as an OMT(\mathcal{LRA}) problem,
 176 we can readily use the same OMT solver used for inference
 177 to also solve the separation oracle. This can be achieved by
 178 selecting a loss function such as the Hamming loss in fea-
 179 ture space:

$$\Delta(\mathbf{I}, \mathcal{O}, \mathcal{O}') := \|\psi(\mathbf{I}, \mathcal{O}) - \psi(\mathbf{I}, \mathcal{O}')\|_1$$

182 This loss function is piecewise-linear, and as such satisfies
 183 the desideratum. While this is the loss which was used in
 184 all experiments, LMT can work with any loss function that
 185 can be encoded as an SMT formula.

3. A Stairway to Heaven

189 We show the potential of the LMT framework on a toy con-
 190 structive problem which consists in learning how to assem-
 191 ble different kinds of *stairways* from examples. A stairway
 192 is made up of a collection of m blocks (rectangles) in a
 193 2D unit-sized bounding box. Figure 1 shows examples of
 194 different types of stairways (a-c), varying in terms of *orien-*
 195 *tation* and *proportions* between block heights and widths,
 196 and a configuration which is not a stairway (d).



199 Figure 1. (a) A *left ladder* 2-stairway. (b) A *right ladder* 2-
 200 stairway. (c) A *right pillars* 2-stairway. (d) not a stairway.

208 Each block is identified by a tuple (x, y, dx, dy) , consisting
 209 of bottom-left coordinates, width and height. An output \mathcal{O}
 210 is given by the set of tuples identifying the blocks. Input
 211 is assumed to be empty in the following, but non-empty
 212 inputs can be used to model for instance partially observed
 213 scenes.

214 Learning amounts to assigning appropriate weights to a set
 215 of soft-constraints in order to maximize the score of stair-
 216 ways of the required type, with respect to non-stairways

¹<http://optimathsat.disi.unitn.it/>

and stairways of other types. Inference amounts to generating configurations (i.e. variables assignments to all blocks) with maximal score, and can be easily encoded as an OMT($\mathcal{LR.A}$) problem. As a first step, we define a set of hard rules to constrain the space of admissible block assignments. We require that all blocks fall within the bounding box, and that blocks do not overlap:

$$\begin{aligned} \forall i \quad & 0 \leq x_i, dx_i, y_i, dy_i \leq 1 \\ \forall i \quad & 0 \leq (x_i + dx_i) \leq 1 \wedge 0 \leq (y_i + dy_i) \leq 1 \\ \forall i \neq j \quad & (x_i + dx_i \leq x_j) \vee (x_j + dx_j \leq x_i) \vee \\ & (y_i + dy_i \leq y_j) \vee (y_j + dy_j \leq y_i) \end{aligned}$$

Furthermore, we require (without loss of generality) blocks to be ordered from left to right, $\forall i \ x_i \leq x_{i+1}$. Note that stairway conditions (apart from non-overlap between blocks) are not included in these hard rules and will be implicitly modelled as soft constraints. To this aim we introduce a set of useful predicates. We use four predicates to encode the fact that a block i may touch one of the four corners of the bounding box, e.g.:

$$\text{bottom_right}(i) := (x_i + dx_i) = 1 \wedge y_i = 0$$

We also define predicates to describe the relative positions of two blocks i and j , such as $\text{left}(i, j)$:

$$\text{left}(i, j) := (x_i + dx_i) = x_j \wedge ((y_j \leq y_i \leq y_j + dy_j) \vee (y_j \leq y_i + dy_i \leq y_j + dy_j))$$

that encodes the fact that block i is touching block j from the left. Similarly, we also define $\text{below}(i, j)$ and $\text{over}(i, j)$. Finally, we combine the above predicates to define the concept of *step*:

$$\text{left_step}(i, j) := (\text{left}(i, j) \wedge (y_i + dy_i) > (y_j + dy_j)) \vee (\text{over}(i, j) \wedge (x_i + dx_i) < (x_j + dx_j))$$

We define $\text{right_step}(i, j)$ in the same manner. This background knowledge allows to encode the property of being a stairway of a certain type as a conjunction of predicates. However, our inference procedure does not have access to this knowledge. We rather encode an appropriate set of *soft rules* (costs) which, along with the associated weights, should bias the optimization towards block assignments that form a stairway of the correct type. Our cost model is based on the observation that it is possible to discriminate between the different stairway types using only four factors: minimum and maximum step size, and amount of horizontal and vertical *material*. For instance, in the cost we account for both the maximum step height of all left steps (a good stairway should not have too high steps) and the minimum step width of all right steps (good

stairways should have sufficiently large steps):

$$\begin{aligned} \text{maxshl} &= m \times \max_{i \in [1, m-1]} \begin{cases} (y_i + dy_i) - (y_{i+1} + dy_{i+1}) & \text{if } i, i+1 \text{ form a left step} \\ 1 & \text{otherwise} \end{cases} \\ \text{minswr} &= m \times \min_{i \in [1, m-1]} \begin{cases} (x_{i+1} + dx_{i+1}) - (x_i + dx_i) & \text{if } i, i+1 \text{ form a right step} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The value of these costs depends on whether a pair of blocks actually forms a left step, a right step, or no step at all. Finally, we write the average amount of vertical material as $\text{vmat} = \frac{1}{m} \sum_i dy_i$. All the other costs can be written similarly. Putting all the pieces together, the complete cost is:

$$\begin{aligned} \text{cost} := & (\text{maxshl}, \text{minshl}, \text{maxshr}, \text{minshr}, \\ & \text{maxswl}, \text{minswl}, \text{maxswr}, \text{minswr}, \\ & \text{vmat}, \text{hmat}) \ \mathbf{w} \end{aligned}$$

The actual weights are learned, allowing the learnt model to reproduce whichever stairway type is present in the training data.

To test the stairway scenario, we generated “perfect” stairways of 2 to 6 blocks for each stairway type to be used as training instances. We then learned a model using all training instances up to a fixed number of blocks (e.g from 2 to 4) and asked the learnt models to generate configurations with a *larger* number of blocks (up to 10) than those in the training set. The results can be found in Figure 2.

The experiment shows that LMT is able to solve the stairway construction problem, and can learn appropriate models for all stairway types, as expected. Some imperfections can be seen when the training set is too small (e.g., only two training examples; first row of each table), but already with four training examples the model is able to generate perfect 10-block stairways of the given type, for all types.

4. Conclusions

Albeit simple, the stairway application showcases the ability of LMT to handle learning in hybrid Boolean-numerical domains characterized by complex combinations of hard and soft constraints, whereas other formalisms are not suited for the task². The application is a simple instance

²The stairway problem can be easily encoded in the Chuch probabilistic programming language. However, the sampling strategies used for inference are not conceived for performing optimization with hard continuous constraints. Even the simple task of generating two blocks, conditioned on the fact that they form a step, is prohibitively expensive. A more comprehensive discussion on alternative approaches and their limitations can be found in the journal version of this work.

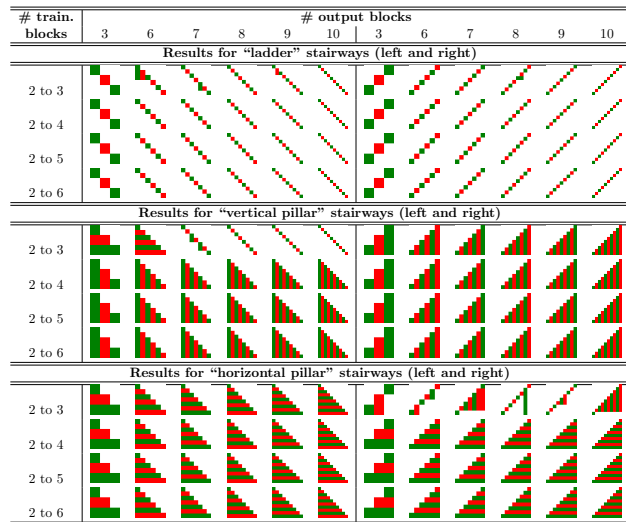


Figure 2. Results for the stairway construction problem. From top to bottom: results for the *ladder*, *horizontal pillar*, and *vertical pillar* cases. Number of blocks in training and generated images are reported in rows and columns respectively.

of a layout problem, where the task is to find an optimal layout subject to a set of constraints. Automated or interactive layout synthesis has a broad range of potential applications, including urban pattern layout (Yang et al., 2013), decorative mosaics (Hausner, 2001) and furniture arrangement (Yu et al., 2011). Note that many spatial constraints can be encoded in terms of relationships between blocks (Yu et al., 2011). Existing approaches typically design an energy function to be minimized by stochastic search. Our approach suggests how to automatically identify the relevant constraints and their respective weights, and can accommodate hard constraints and exact search. This is especially relevant for *water-tight* layouts (Peng et al., 2014), where the whole space needs to be filled (i.e. no gaps or overlaps) by deforming basic elements from a pre-determined set of templates (as in residential building layout (Merrell et al., 2010)). Generally speaking, the LMT framework allows to introduce a learning stage in all application domains where SMT and OMT approaches have shown their potential, ranging, e.g., from engineering of chemical reactions (Fagerberg et al., 2012) to synthetic biology (Yordanov et al., 2013).

References

Barrett, C., Sebastiani, R., Seshia, S. A., and Tinelli, C. Satisfiability modulo theories. In *Handbook of Satisfiability*, chapter 26, pp. 825–885. IOS Press, 2009.

Choi, Jaesik and Amir, Eyal. Lifted relational variational inference. In *UAI'12*, pp. 196–206, 2012.

Fagerberg, R., Flamm, C., Merkle, D., and Peters, P. Exploring chemistry using smt. In *CP'12*, pp. 900–915, 2012.

Goodman, Noah D., Mansinghka, Vikash K., Roy, Daniel M., Bonawitz, Keith, and Tenenbaum, Joshua B. Church: a language for generative models. In *UAI*, pp. 220–229, 2008.

Gutmann, B., Jaeger, M., and De Raedt, L. Extending problog with continuous distributions. In *Inductive Logic Programming*, volume 6489, pp. 76–91. 2011.

Hausner, Alejo. Simulating decorative mosaics. In *SIG-GRAPH '01*, pp. 573–580, 2001.

Islam, M. A., Ramakrishnan, C. R., and Ramakrishnan, I. V. Inference in probabilistic logic programs with continuous random variables. *Theory Pract. Log. Program.*, 12(4-5):505–523, September 2012. ISSN 1471-0684.

Merrell, Paul, Schkufza, Eric, and Koltun, Vladlen. Computer-generated residential building layouts. *ACM Trans. Graph.*, 29(6):181:1–181:12, December 2010.

Närman, P., Buschle, M., König, J., and Johnson, P. Hybrid probabilistic relational models for system quality analysis. In *EDOC*, pp. 57–66. IEEE Computer Society, 2010.

Peng, Chi-Han, Yang, Yong-Liang, and Wonka, Peter. Computing layouts with deformable templates. *ACM Trans. Graph.*, 33(4):99:1–99:11, July 2014.

Sebastiani, Roberto and Tomasi, Silvia. Optimization Modulo Theories with Linear Rational Costs. *ACM Transactions on Computational Logics*, 16, 2015.

Tsochantaridis, Ioannis, Joachims, Thorsten, Hofmann, Thomas, and Altun, Yasemin. Large margin methods for structured and interdependent output variables. *J. Mach. Learn. Res.*, 6:1453–1484, December 2005.

Wang, Jue and Domingos, Pedro. Hybrid markov logic networks. In *AAAI'08*, pp. 1106–1111, 2008.

Yang, Yong-Liang, Wang, Jun, Vouga, Etienne, and Wonka, Peter. Urban pattern: Layout design by hierarchical domain splitting. *ACM Trans. Graph.*, 32(6):181:1–181:12, November 2013.

Yordanov, Boyan, Wintersteiger, Christoph M., Hamadi, Youssef, and Kugler, Hillel. Smt-based analysis of biological computation. In *NASA Formal Methods Symposium 2013*, pp. 78–92, May 2013.

Yu, Lap-Fai, Yeung, Sai-Kit, Tang, Chi-Keung, Terzopoulos, Demetri, Chan, Tony F., and Osher, Stanley J. Make it home: Automatic optimization of furniture arrangement. *ACM Trans. Graph.*, 30(4):86:1–86:12, July 2011.